SOLUTIONS FOR ADMISSIONS TEST IN MATHEMATICS, JOINT SCHOOLS AND COMPUTER SCIENCE
WEDNESDAY 5 NOVEMBER 2008

Mark Scheme:

Each part of Question 1 is worth four marks which are awarded solely for the correct answer.

Each of Questions 2-7 is worth 15 marks

QUESTION 1:

A. As $y = 2x^3 - 6x^2 + 5x - 7$ then

$$y' = 6x^2 - 12x + 5.$$

The quadratic y' has discriminant $12^2 - 4 \times 6 \times 5 = 24 > 0$ and hence the equation $y' = 0$ has two distinct real roots. **The answer is (c).**

B. As $\pi < 10$ then

$$L = \log_{10} \pi < 1.$$

So

$$\sqrt{\log_{10}(\pi^2)} = \sqrt{2L} > \sqrt{L \times L} = L; \qquad \left(\frac{1}{\log_{10}\pi}\right)^3 = L^{-3} > 1; \qquad \frac{1}{\log_{10}\sqrt{\pi}} = \frac{2}{L} > 2.$$

The answer is (a).

C. We will write $c = \cos\theta$ and $s = \sin\theta$ for ease of notation. Eliminating y from the simultaneous equations

$$cx - sy = 2, \qquad sx + cy = 1;$$

we get

$$2c + s = c(cx - sy) + s(sx + cy) = \left(c^2 + s^2\right)x = x$$

and similarly eliminating x we find

$$c - 2s = (-s)(cx - sy) + c(sx + cy) = \left(s^2 + c^2\right)y = y.$$

Hence the equations care solvable for any value of θ. **The answer is (a).**

D. By the remainder theorem when a polynomial $p(x)$ is divided by $x - 1$ then the remainder is $p(1)$. So the required remainder here is

$$1 + 3 + 5 + 7 + \cdots + 99 = \frac{50}{2}(1 + 99) = 2500$$

as the series is an arithmetic progression. **The answer is (b).**

E. The highest power of x in $(2x^6+7)^3$ is x^{18} and in $(3x^8-12)^4$ is x^{32} so the highest power in $[\cdots]^5$ is $(x^{32})^5 = x^{160}$. The highest power of x in $(3x^5-12x^2)^5$ is x^{25} and in $(x^7+6)^4$ is x^{28}, so that the highest power of x in $[\ldots]^6$ is $(x^{28})^6 = x^{168}$. Thus the highest power of x in $\{\ldots\}^3$ is $(x^{168})^3 = x^{504}$. **The answer is (d).**

F. Suppose that, when the trapezium rule is used to estimate the integral $\int_0^1 f(x)\,dx$, an overestimate of E is produced. If the same number of intervals are used in the following calculations then:

(a) to estimate $\int_0^1 2f(x)\,dx$ an overestimate of $2E$ will be produced, as the relevant graphs have been stretched by a factor of 2 and all areas doubled;

(b) to estimate $\int_0^1 (f(x)-1)\,dx$ an overestimate of E will be produced, as the relevant graphs have been translated down by 1 and all areas remain the same;

(c) to estimate $\int_1^2 f(x-1)\,dx$ an overestimate of E will be produced, as the relevant graphs have been translate right by 1 and all areas remain the same;

(d) to estimate $\int_0^1 (1-f(x))\,dx$ an underestimate of E will be produced, as the relevant graphs have been reflected in the x-axis – turning the overestimate to an underestimate – and translated up by 1, which changes nothing with regard to areas. **The answer is (d).**

G. As $4x-x^2-5 = -(x-2)^2-1$, then $y=(4x-x^2-5)^{-1}$ is always negative and has a minimum value at $x=2$. **The answer is (c).**

H. If we set $y=3^x$ then the equation $9^x - 3^{x+1} = k$ now reads

$$y^2 - 3y - k = 0.$$

This has solutions

$$y = \frac{3 \pm \sqrt{9+4k}}{2}$$

which are real when $k \geqslant -9/4$. As $y=3^x$ then we further need that $y>0$ for x to be real, but this is not a problem as the larger root is clearly positive. **The answer is (a).**

I. We have

$$S(1)+S(2)+S(3)+\cdots+S(99) = S(00)+S(01)+\cdots+S(99)$$

and in the 100 two-digit numbers $00,\ldots,99$ there are twenty $0s$, twenty $1s$, ..., twenty 9s. So

$$S(1)+S(2)+S(3)+\cdots+S(99) = 20\times(0+1+\cdots+9) = 20\times\frac{10}{2}(0+9) = 900$$

and **the answer is (c).**

J. Note that

$$(3+\cos x)^2 \geqslant (3-1)^2 = 4; \qquad 4-2\sin^8 x \leqslant 4.$$

So the equation will hold only when $\cos x = -1$ and $\sin x = 0$. In the range $0 \leqslant x < 2\pi$ this only occurs at $x=\pi$. **The answer is (b).**

2. (i) [2 marks] A fairly obvious pair (x_1, y_1) that satisfy $(x_1)^2 - 2(y_1)^2 = 1$ is $x_1 = 3$ and $y_1 = 2$.

(ii) [6 marks] Note

$$\begin{aligned} (x_{n+1})^2 - 2(y_{n+1})^2 &= (x_n)^2 - 2(y_n)^2 \\ \iff (3x_n + 4y_n)^2 - 2(ax_n + by_n)^2 &= (x_n)^2 - 2(y_n)^2 \\ \iff \left(8 - 2a^2\right)(x_n)^2 + (24 - 4ab)\, x_n y_n + \left(18 - 2b^2\right)(y_n)^2 &= 0 \end{aligned}$$

In order to have $(x_{n+1})^2 - 2(y_{n+1})^2 = (x_n)^2 - 2(y_n)^2$ we need

$$2a^2 = 8, \qquad 4ab = 24, \qquad 2b^2 = 18.$$

We further require that $a, b > 0$. We see that $a = 2$ and $b = 3$ solve all three equations.

(iii) [4 marks] Starting with $x_1 = 3$, $y_1 = 2$ we find:

$$\begin{aligned} x_1 &= 3, & y_1 &= 2; \\ x_2 &= 3 \times 3 + 4 \times 2 = 17, & y_2 &= 2 \times 3 + 3 \times 2 = 12; \\ x_3 &= 3 \times 17 + 4 \times 12 = 99, & y_3 &= 2 \times 17 + 3 \times 12 = 70. \end{aligned}$$

So $X = 99$ and $Y = 70$ is such a pair.

(iv) [3 marks] For the generated sequences, $(x_n), (y_n)$, we have

$$(x_n)^2 - 2(y_n)^2 = 1 \quad \text{for each } n.$$

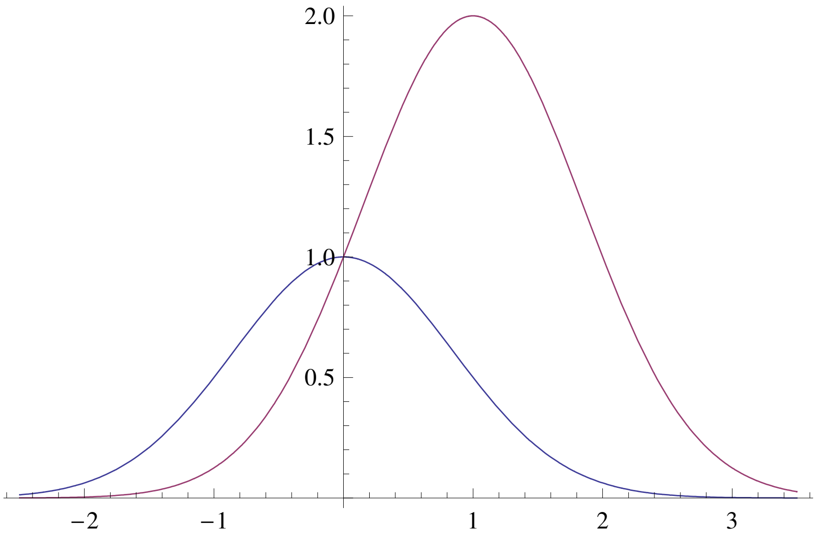
Also the integers x_n and y_n are getting increasingly larger because of how they are defined in (ii). So

$$\left(\frac{x_n}{y_n}\right)^2 - 2 = \frac{1}{(y_n)^2} \approx 0 \quad \text{for large } n,$$

and $x_n / y_n \approx \sqrt{2}$ as x_n and y_n are both positive.

3. (i) [3 marks] (A) is $-f(x)$; (B) is $f(-x)$; (C) is $f(x-1)$.

(ii) [9 marks] As $2^{2x-x^2} = 2 \times 2^{-(x-1)^2}$ then the graph of $y = 2^{2x-x^2}$ is the graph of $y = 2^{-x^2}$ translated to the right by 1 and stretched parallel to the y-axis by a factor of 2.



Plots of $y = 2^{-x^2}$ and $y = 2^{2x-x^2}$.

(iii) [3 marks] $c = \frac{1}{2}$. The graph of $2^{-(x-c)^2}$ is the graph of 2^{-x^2} translated c to the right. The integral $I(c)$ represents the area under the graph between $0 \leqslant x \leqslant 1$. As the graph is symmetric/even and decreasing away from 0 then this area is maximised by having the apex half way along the interval $0 \leqslant x \leqslant 1$, i.e. at $x = 1/2$ which occurs when $c = \frac{1}{2}$.

4. (i) [4 marks] We can complete the squares in $x^2 - px + y^2 - qy = 0$ to get

$$\left(x - \frac{p}{2}\right)^2 + \left(y - \frac{q}{2}\right)^2 = \frac{p^2+q^2}{4} \tag{1}$$

which is the equation of the circle with centre: $(p/2, q/2)$ and area: $\pi\left(p^2+q^2\right)/4$. Either by checking the original question, or the rearranged one, we can see that

$$x^2 - px + y^2 - qy = \begin{cases} 0 & \text{at } (0,0)\,, \\ p^2 - p^2 + 0 = 0 & \text{at } (p,0)\,, \\ 0 + q^2 - q^2 = 0 & \text{at } (0,q)\,. \end{cases}$$

(ii) [5 marks] The area of OPQ is $pq/2$. So

$$\frac{\text{area of circle } C}{\text{area of triangle } OPQ} = \left(\frac{\pi\left(p^2+q^2\right)}{4}\right) \Big/ \left(\frac{1}{2}pq\right) = \frac{\pi\left(p^2+q^2\right)}{2pq}.$$

Note

$$\frac{\pi\left(p^2+q^2\right)}{2pq} \geqslant \pi \iff p^2+q^2 \geqslant 2pq \iff (p-q)^2 \geqslant 0,$$

proving the required inequality.

(iii) [6 marks] Rearranging

$$\frac{\pi\left(p^2+q^2\right)}{2pq} = 2\pi \iff p^2+q^2 = 4pq \iff \left(\frac{p}{q}\right)^2 - 4\left(\frac{p}{q}\right) + 1 = 0,$$

which is a quadratic equation in p/q, and so

$$\frac{p}{q} = \frac{4 \pm \sqrt{16-4}}{2} = 2 \pm \sqrt{3}.$$

Now $p/q = \tan OQP$, $q/p = \tan OPQ$ and so

$$\{\tan OQP, \tan OPQ\} = \left\{2-\sqrt{3}, 2+\sqrt{3}\right\}$$

with the order depending on whether $p < q$ or $p > q$.

[It happens that $\arctan\left(2-\sqrt{3}\right) = \pi/12$ and $\arctan\left(2+\sqrt{3}\right) = 5\pi/12$, but appreciation of this was not expected.]

5. (i) [3 marks] After the first/second/third students have gone by the doors look like:

Locker	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Student 1	O	O	O	O	O	O	O	O	O	O	O	O	O	O
Student 2	O	C	O	C	O	C	O	C	O	C	O	C	O	C
Student 3	O	C	C	C	O	O	O	C	C	C	O	O	O	C

We can see that the lockers now repeat in a pattern OCCCOO every 6 lockers. As $1000 = 166 \times 6 + 4$ we have 166 repeats of this pattern and 4 remaining lockers that go OCCC. So there are $166 \times 3 = 498$ closed lockers amongst the complete cycles and 3 further in the incomplete cycle. That is, there are 501 closed lockers in all.

(ii) [4 marks] After the fourth student has gone by we have the following:

Locker	1	2	3	4	5	6	7	8	9	10	11	12	13	14
Student 3	O	C	C	C	O	O	O	C	C	C	O	O	O	C
Student 4	O	C	C	O	O	O	O	O	C	C	O	C	O	C

with the pattern repeating every 12 lockers in the form OCCOOOOOCCOC. Each cycle contains 5 closed and 7 open doors. Now $1000 = 83 \times 12 + 4$ and so we have $83 \times 5 = 415$ closed lockers amongst the complete cycles and 2 further amongst the incomplete cycle OCCO. In all then there are 417 closed lockers.

(iii) [4 marks] Locker 100 starts off closed (as all lockers do) and then its state is altered by every nth student where n is a factor of 100, i.e. by students $1, 2, 4, 5, 10, 20, 25, 50, 100$. So 9 students change the state and as this is odd then overall the state will have been changed to open.

(iv) [4 marks] Locker 1000 starts off closed (as all lockers do) and then its state is altered by every nth student where n divides 1000 and $n \leqslant 100$, i.e. by $1, 2, 4, 5, 8, 10, 20, 25, 40, 50, 100$. So 11 students change the door's state and as this is odd then overall the state will again have been changed to open.

6. (i) [5 marks] We have six possibilities:

A-B-C = St-L-Sw, St-Sw-L, L-St-Sw, L-Sw-St, Sw-L-St, Sw-St-L.

The statement "I am the liar" cannot be made by St or L; this excludes the first four possibilities above.

The second statement "A is the liar" excludes Sw-St-L and so we are left with Sw-L-St. Answer: B is the Liar.

(The third statement is not actually needs but doesn't contradict the Sw-L-St arrangement.)

(ii) [5 marks] We have six possibilities:

P-Q-R = S-L-C, S-C-L, L-S-C, L-C-S, C-L-S, C-S-L.

One of these statements is from a saint and so true. This means that the Liar has to follow the Saint in cyclic order and this means the only remaining possibilities are

P-Q-R = S-L-C, L-C-S, C-S-L.

In the first two cases the Contrarian follows the Liar and so tells the truth. But this contradicts the actual statements so the only possibility remaining is C-S-L. Answer: R is the Liar.

(iii) [5 marks] We have six possibilities:

X-Y-Z = S-L-C, S-C-L, L-S-C, L-C-S, C-L-S, C-S-L.

We will take these case by case:

- S-L-C: As the Contrarian is following the Liar, statement 3 had to be true but isn't in this case.
- L-C-S: As the Contrarian is following the Liar, statement 2 had to be true but isn't in this case.
- C-S-L: As the Contrarian is following the Liar, statement 4 had to be true but isn't in this case.
- S-C-L: In this case, statement 4 is a lie and so the Contrarian would tell the truth in Statement 5 but doesn't.
- C-L-S: The Contrarian tells the truth to begin contradicting his nature.
- L-S-C: This is the only remaining case and is consistent.

Answer: X is the Liar.

7. (i) [3 marks] The empty word has zero length which is even. If a new word is formed by Rule 2 then aWb will have the same parity of length as W had. Also if U and V are even-length words then so will be UV. So new words formed from words of even length will themselves be even.

(ii) [5 marks]
Length 0 words: $\varnothing$.
Length 2 words: *ab*.
Length 4 words: *abab*, *aabb*
Length 6 words: *ababab*, *abaabb*, *aabbab*, *aababb*, *aaabbb*

(iii) [3 marks] In $\varnothing$ there are the same number of as and bs, namely none. If W has the same number then so will aWb, formed by Rule 2. Also if U and V each have the same number of as and bs then so will UV. So new words formed by Rules 2 and 3 always have the same property.

(iv) [4 marks] A word of the form $aWbW'$ will be of length $2n+2$ if

$$\text{length}\,(W) + \text{length}\,(W') = 2n.$$

So if W has length $2k \leqslant 2n$ then W' has length $2\,(n-k)$. There are C_k words of the former length and C_{n-k} of the latter length. So we may generate $C_k C_{n-k}$ such words of length $2n+2$ in this manner for each k. That is,

$$\sum_{k=0}^{n} C_k C_{n-k}$$

in all. Further, because the uniqueness of form in the given hint, all words of length $2n+2$ are counted amongst these words and none are doubly counted. That is

$$C_{n+1} = \sum_{k=0}^{n} C_k C_{n-k}$$